McBilliards Documentation: The Word Window

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1 Introduction

McBilliards is a java-based program whose purpose is to investigate periodic billiard paths in triangles. This program has a certain amount of documentation built into it. While the built-in documentation does explain some of the mathematics behind McBilliards, its main purpose is to explain how to actually operate the program. The purpose of these notes, on the other hand, is to delve more deeply into the mathematics behind McBilliards. In this first section we will try to give some basic organizational information about the notes.

Relevant McBilliards Windows: word window; unfolding window

Topics Covered:
hexpath
squarepath
dart decomposition
spines

Dependence on prior notes: the *basic notions* packet

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2 Review of the Basic Notions Packet

For convenience, we quickly summarize the material covered in the *basic notions* packet.

Parameter Space: The parameter space Δ in McBilliards is the unit square $(0,1)^2$. The point point (x_1, x_2) corresponds to the triangle, two of whose angles are $\pi x_1/2$ and $\pi x_2/2$. The *obtuse region* consists in the subset where $x_1 + x_2 < 1$.

Combinatorial Types and Orbit Tiles: To each infinite periodic word W, with digits in the set $\{1,2,3\}$, we assign the region $O(W) \subset \Delta$ as follows: A point belongs to $O(W)$ if W describes the combinatorics of a periodic billiard path in the corresponding triangle. By this we mean that we label the sides of the triangle 1, 2, and 3, and then read off W as the sequence of successive edges encountered by the billiard path. We call $O(W)$ an *orbit tile* and W a *combinatorial type*. (We sometimes call W a *word* as well.) McBilliards only searches for combinatorial types having even length.

Unfoldings: The *unfolding* of a word W with respect to a triangle T, which we denote by $U(W,T)$, is the union of triangles obtained by reflecting T out according to the digits of W. This is a classic construction, treated by many authors. We will often use a variant of our notatiuon. A point X in parameter space represents a triangle $T = T_X$. We will often write $U(W,X)$ in place of $U(W,T)$.

Stability: A word W is called *stable* if the first and last sides of $U(W,T)$ are parallel for any triangle T. This implies that $O(W)$ is an open set. McBilliards only searches for stable words. We can break W into couplets—e.g.

23 23 13 23 13 12 32 32 31 31 31

Let (ij) denote the number of times the couplet ij appears. For instance $(23) = 3$. Next, we define $((ij)) = (ij) - (ji)$. For instance $((23)) = 3 - 2 = 1$. Then W is stable if and only if $((12)) = ((23)) = ((31))$. The example above is a stable word.

3 The Hexpath

Now we start the new material. The building block of our construction is a centrally symmetric hexagon whose opposite sides are labelled 1, 2, and 3. McBilliards concentrates on two different kinds of these hexagons:

1. The regular hexagon.

2. The *square-biased hexagon*. This hexagon has a wierd shape, but the hexagon midscribed in it has the integral coordinates shown on the left hand side of Figure 1.

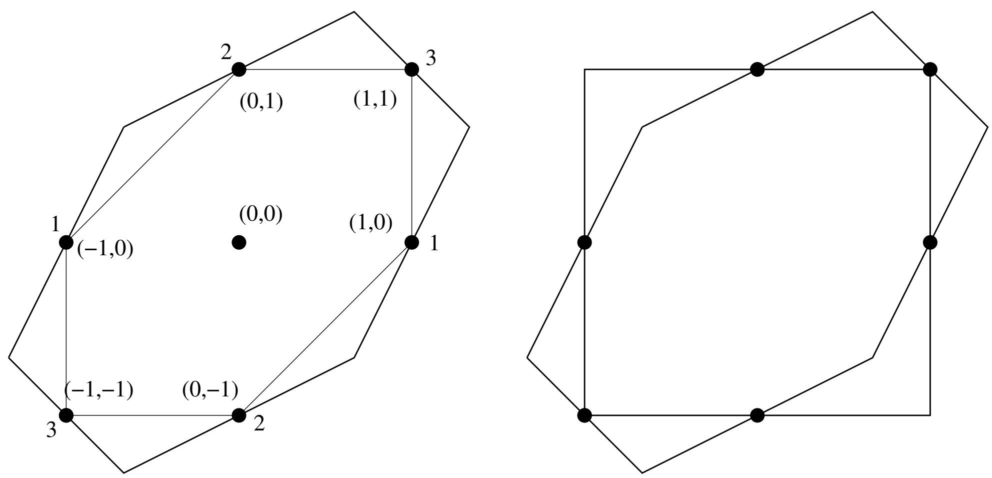

Figure 1: The square biased hexagon

Given any centrally symmetric hexagon as our starting point, we let $\mathcal{H}$ denote the tiling of the plane by translates of the hexagon. We are only interested in the edges of $\mathcal{H}$. Obviously, we get the most symmetric version of $\mathcal{H}$ when we start with the regular hexagon. However, when we start with the square-biased hexagon, the midpoints of $\mathcal{H}$ are labelled by integer coordinates, and indeed there is a natural square grid whose edges have the same midpoints. The right hand side of Figure 1 suggests the relation between $\mathcal{H}$ and a square grid in this case.

McBilliards makes its constructions based either on the regular hexagon or the square-biased hexagon. There is a toggle switch on the *word window* which lets you change back and forth. In these notes we will draw all our pictures using the square-biased hexagon as our starting point.

Given the word W, we can draw a path in $\mathcal{H}$ by following the edges as determined by the word: we move along the dth family when we encounter the digit d. Our convention is that the long sides of our triangle are labelled by 3. Figure 2.3 shows the path corresponding to the example given in Figure 2.1.

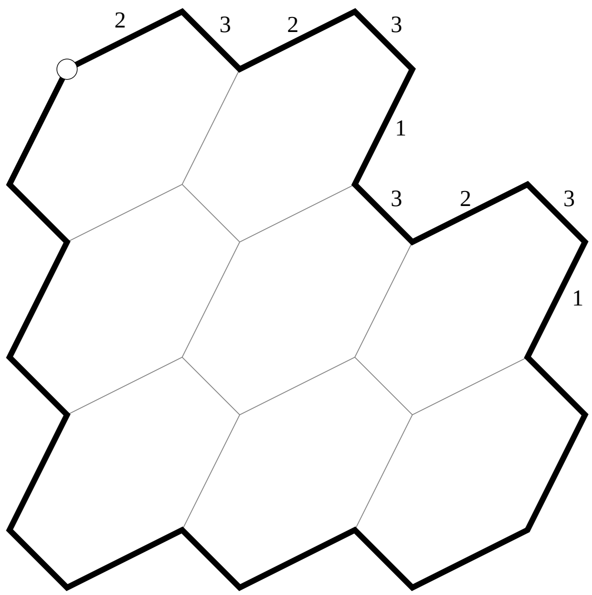

Figure 2: The hexpath for $W = 23231323131232323131 31$.

The dot in the picture indicates the start of the path. We call this path the *hexpath* and denote it by $H(W)$. We can reformulate stability in terms of the hexpath. The even length word W is stable iff $H(W)$ is a closed path. This is a fairly immediate consequence of the stability criterion, and we leave the proof to the reader.

$H(W)$ has a nice topological interpretation. Let Σ denote the flat 3-punctured sphere obtained by doubling the triangle T and deleting the vertices. Σ deformation retracts to a θ-graph Θ, a graph with two vertices each having valence 3. If we draw Θ symmetrically as the double of a piecewise geodesic Y-graph, and pick the shape of $\mathcal{H}$ correctly, then $\mathcal{H}$ is naturally the universal abelian cover of Θ. If W describes a periodic billiard path on T then this billiard path can be interpreted as a closed geodesic on Σ. We can homotope this path so that it is a subset of Θ. This Θ-path lifts to a closed loop in $\mathcal{H}$ iff W is stable, iff W lies in the commutator subgroup of $\pi_1(\Sigma)$. In this case, the lifted loop is precisely $H(W)$.

4 The Squarepath

It turns out that the hexpath $H(W)$ contains precisely the same information as a certain rectilinear path, which we call the *squarepath*. Each vertex of the hexpath has a unique type 3 edge emanating from it. The squarepath is obtained by connecting the midpoints of these type-3 edges together, in order. We denote the squarepath by $\widehat{Q}(W)$. We can also define similar paths based on the edges of type 1 or 2. These paths are somewhat more complicated, though they will be of theoretical importance for us. In practice, however, we will always try to work with the type 3 edges.

If we mark off points on the squarepath at integer steps (starting with a vertex) the resulting points are naturally in bijection with the type 3 edges of the unfolding. In the next section we will elaborate on this bijection. Figure 2.4 shows the squarepath for the examples we have been considering The hexpath is drawn underneath in grey.

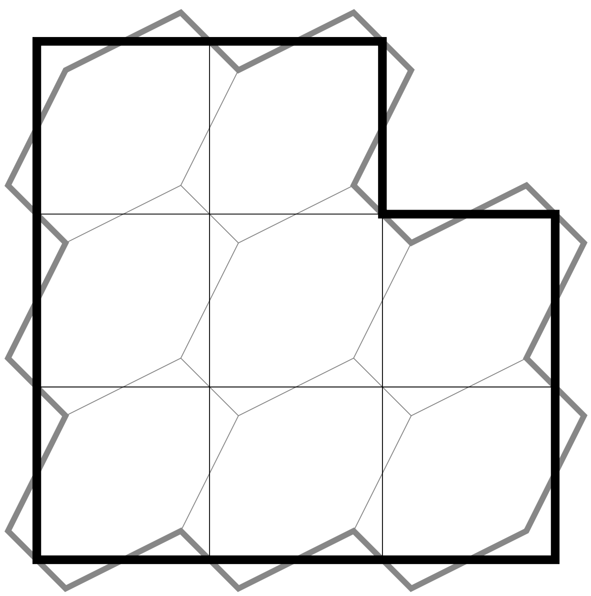

Figure 3: $\widehat{Q}(W)$ in black and $H(W)$ in grey.

It is possible to reconstruct $H(W)$ from $\widehat{Q}(Q)$, when $\widehat{Q}(W)$ is a closed loop. When $\widehat{Q}(W)$ is embedded, this loop bounds a finite union of squares. We simply replace each square by the associated hexagon. Then $H(W)$ is the boundary of the union of hexagons. In general, $H(W)$ is the union of all the edges of $\mathcal{H}$ which intersect $\widehat{Q}(W)$. There is a natural ordering to these edges and the result is a closed loop.

5 Dart Decomposition

We call a word W *obtuse* if $O(W)$ lies in the region of Δ consisting of obtuse triangles. (Pat Hooper has proved that $O(W)$ either lies in the obtuse region of Δ or is disjoint from the obtuse region.) If W is obtuse, then W cannot contain the strings 121 or 212. When W is obtuse, there is a simple algorithm for deducing the combinatorics of $U(W,T)$ from the square path. In this section we describe this algorithm.

k-dart is a union of k isosceles triangles, arranged around a common vertex, in the pattern shown in Figure 2.5 for $k = 2,3,4..$ A k-dart is just an unfolding with respect to either the word $(13)^{k-1}1$ or the word $(23)^{k-1}2$.

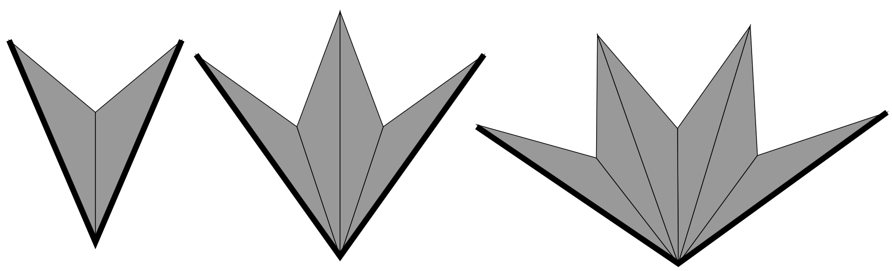

Figure 4: k-darts for $k = 1,2,3$

We say that the 3-*spine* of the dart is the union of the two outermost long edges. We have highlighted the spines of our darts in Figure 4.

The relation of $U(W,*)$ to $\widehat{Q}(W)$ is as follows:

- The maximal darts of the unfolding are in bijection with the edges of the square path. (The maximal k-darts correspond to edges of length $2k$.) The maximal darts are glued together along their 3-spines.

- Two consecutive maximal darts lie on opposite sides of their common 3-edge iff $\widehat{Q}(W)$ makes a northwest or southeast turn at the vertex corresponding to this 3-edge.

To make this work precisely, we need to take the infinite periodic continuation of U, or else identify the first and last sides of U to make an annulus. As it is, the reader needs to take special care in figuring out how the rightmost maximal dart fits together with the leftmost one. Figure 5 shows a picture of $U(W,(2/5,1/3))$, where W is the example above. We have highlighted the spines of the maximal darts.

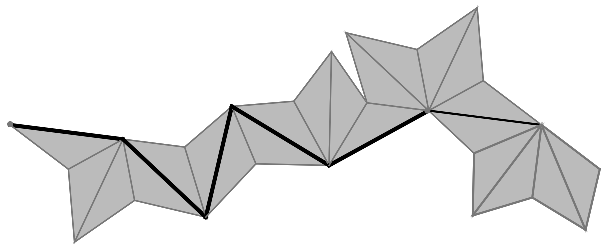

Figure 5: Dividing the unfolding into maximal darts.

6 Spines and Polygonal Paths

Let $U = U(W,T)$. Let $\widetilde{U}$ be the infinite periodic continuation of U. There is a unique infinite polygonal path in $\widetilde{U}$ consisting of the type-d edges of the unfolding. The image of this path in U is what we call the *d-spine*. When W is an obtuse word, the 3-spine of U is precisely the union of the 3-spines of the maximal darts comprising U. Thus, Figure 5 highlights the 3-spine of U.

Say that an edge of U is *reflective* if this edge is an edge of two consecutive triangles of U. In other words, an edge is reflective if, at some point during the creation of U, we reflect across this edge. The reflective edges of U are naturally in bijection with the edges of $H(W)$. The vertices of $H(W)$ are naturally in bijection with the triangles of U. Each vertex of $H(W)$ is incident to 3 edges of $\mathcal{H}$. These 3 edges are naturally bijective with the 3 sides of the corresponding triangle in U.

There is a natural linear ordering to the type-d edges of U. From the discussion in the previous paragraph, the union of type-d edges of U naturally defines a sequence $e_1, e_2, e_3, ...$ of edges of $\mathcal{H}$. From this sequence of edges we create a sequence of points by considering $m_1 m_{,2} m_{,3} ...$ Here m_j is the midpoint of e_j. The polygonal path connecting the points $m_1, m_2, m_3 ...$ is what we call the d-path. We denote it by $\widehat{Q}_d(W)$. If we work with the periodic continuation $\widetilde{U}$ then the corresponding polygonal path traces endlessly around the same polygon. Thus, is is sometimes useful for us to consider $\widetilde{Q}_d$ as a closed polygon.

The polygon $\widetilde{Q}_3(W)$ is precisely the squarepath. We do not specially name the other two paths $\widetilde{Q}_1(W)$ and $\widetilde{Q}_2(W)$ because they are less useful to us. However, the *word window* in McBilliards allows the user to see these paths as well.

How is the d-path related to the d-spine? Recall that the d-spine is a (typically proper) subset of the union of all the type-d edges. Thus, the vertices of $\widetilde{Q}_d$ corresponding to the d-spine ought to be some distinguished subset. The beautiful answer is that the *genuine vertices* of $\widetilde{Q}_d$ are the ones which correspond to the edges of the d-spine. By *genuine vertex* we mean a vertex at which the d-path actually makes a bend. For instance, the squarepath in Figure 3 has 6 genuine vertices, corresponding to the 6 black edges in Figure 5.

We can phrase the content of the preceding paragraph in a different way. As an alternate definition of $\widetilde{Q}_d$, we could say simply that its consecutive vertices are the midpoints of the edges of $\mathcal{H}$ which correspond to the edges of the d-spine. This gives the same result as above, except that the *extra vertices* of $\widetilde{Q}_d$ are simply ignored.

It turns out that the geometry of the d-path is closely related to the geometry of the d-spine. This relation is the key to our method for computing the orbit tile $O(W)$. We will explain this in a later note packet.