McBilliards Documentation: Basic Notions

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1 Introduction

McBilliards is a java-based program whose purpose is to investigate periodic billiard paths in triangles. This program has a certain amount of documentation built into it. While the built-in documentation does explain some of the mathematics behind McBilliards, its main purpose is to explain how to actually operate the program. The purpose of these notes, on the other hand, is to delve more deeply into the mathematics behind McBilliards. In this first section we will try to give some basic organizational information about the notes.

Relevant McBilliards Windows: general

Topics Covered:
parameter space
comtinatorial types and orbit tiles
unfoldings
stability

Dependence on prior notes: none

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2 The Parameter Space

The parameter space Δ in McBilliards is the unit square $(0,1)^2$. The point point (x_1,x_2) corresponds to the triangle, two of whose angles are $\pi x_1/2$ and $\pi x_2/2$. Our parametrization is somewhat redundant. For instance, the points (x_1,x_2) and (x_2,x_1) refer to the same geometric triangle, but with the angles labelled differently. Mainly we are interested in the open triangle where $x_1+x_2<1$. This is the region of obtuse triangles.

3 Combinatorial Types and Orbit Tiles

To each infinite periodic word W, with digits in the set $\{1,2,3\}$, we assign the region $O(W)\subset\Delta$ as follows: A point belongs to $O(W)$ if W describes the combinatorics of a periodic billiard path in the corresponding triangle. By this we mean that we label the sides of the triangle 1, 2, and 3, and then read off W as the sequence of successive edges encountered by the billiard path. We call $O(W)$ an *orbit tile* and W a *combinatorial type*. (We sometimes call W a *word* as well.) McBilliards only searches for combinatorial types having even length. A combinatorial type of odd length can simply be doubled, to produce a new combinatorial type of even length. Let $|W|$ denote the length of W.

McBilliards has two basic functions:

- **Searching:** Given a point $X\in\Delta$ and an integer N, find all the combinatorial types W such that $|W|\leq N$ and $X\in O(W)$.

- **Plotting:** Given W plot $O(W)$ to a specified degree of accuracy.

In order to perform these tasks efficiently, and let the user see how things work, McBilliards has many auxiliary features as well.

4 Unfoldings

The *unfolding* of a word W with respect to a triangle T, which we denote by $U(W,T)$, is the union of triangles obtained by reflecting T out according to the digits of W. This is a classic construction, treated by many authors. We will often use a variant of our notatiuon. A point X in parameter space represents a triangle $T = T_X$. We will often write $U(W,X)$ in place of $U(W,T)$.

W represents a periodic billiard path in T iff the first and last sides of $U(W,T)$ are parallel and the interior of $U(W,T)$ contains a line segment L, called a *centerline*, such that L intersects the first and last sides at corresponding points. When the first and last sides are parallel, we rotate so that the translation which identifies these edges is a horizontal translation. In general, we call this translation the *holonomy*

Assuming that $U(W,T)$ has been rotated as above, there is a sequence of vertices which runs across the top of $U(W,T)$. We call these the *top vertices* and often denote them by $a_1, a_2, ...$ going from the left. There is a sequence of vertices which runs across the bottom. We call these *bottom vertices* and denote them by $b_1, b_2, ...$ going from the left. The *reflection edges* of the unfolding are the ones which connect top vertices to bottom vertices.

Figures 1 and 2 shows examples of unfolding for the same word with respect to different points in Δ.

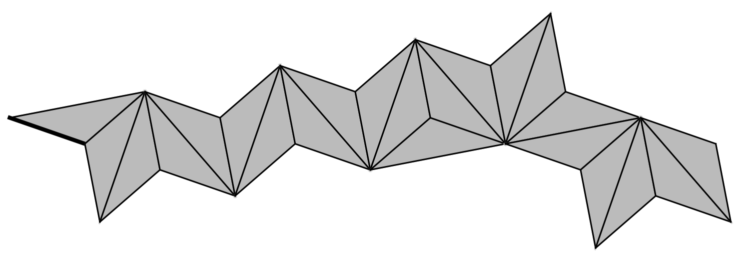

Figure 1.1: $U(23231323131232323131 31,(1/3,1/3))$

It is worth pointing out that two edges in Figure 1 actually coincide. The edges we have in mind are (a_6a_7) and (a_7a_8). In Figure 2, these edges are separated from each other.

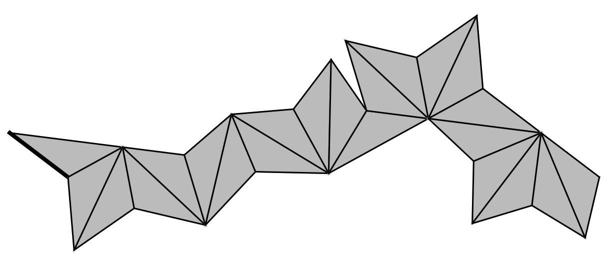

Figure 1.2: $U(2323132313123232313131,(2/5,1/3))$

The first side (a_1b_1) of U has been highlighted in both examples. In the first example U has a centerline, and in the second example U does not. However, in both examples, the first and last sides of U are parallel. We will discuss this general phenomenon of parallelism in the next section.

Figure 1 shows that

$$(1/3, 1/3) \in O(W)$$

and Figure 2 shows that

$$(2/5, 1/3) \not\in O(W).$$

5 Stability

A word W is called *stable* if the first and last sides of $U(W,T)$ are parallel for any triangle T. This implies that $O(W)$ is an open set. McBilliards only searches for stable words. Here we describe a well-known stability criterion for words W which have even length.

We take our word W and break it up into couplets. The example above yields:

23 23 13 23 13 12 32 32 31 31 31

Let (ij) denote the number of times the couplet ij appears. For instance $(23) = 3$. Next, we define $((ij)) = (ij) - (ji)$. For instance $((23)) = 3 - 2 = 1$.

Lemma 5.1 *W is a stable word if and only if* $((12)) = ((23)) = ((31))$.

Proof: We shall prove that W is stable if $(12)) = ((23)) = ((31))$. Our proof can easily be adapted to show the converse. Let A be the common value of $((12))$, etc. Let $2n$ denote the the length of W. Let $\widetilde{U}$ denote the periodic continuation of the unfolding $U(W,T)$. Let T_0 and T_{2n} be two triangles of $\widetilde{U}$ which are $2n$ apart. The triangle T_j and T_{j+2} are related by a rotation by $\pm 2\theta$, where θ is one of the angles of T. The sign and the choice of θ depends on the jth couplet. Here is the dependence:

- $12 \to +2\theta_3$.
- $21 \to -2\theta_3$.
- $23 \to +2\theta_1$.

- $32 \to -2\theta_1$.
- $31 \to +2\theta_2$.
- $13 \to -2\theta_2$.

Thus, the total amount we rotate T_0 to produce T_{2n} is

$$((12)) + ((23)) + ((31)) = A \times (\theta_1 + \theta_2 + \theta_3) = 2\pi N.$$

Then T_0 and T_{2n} are translation equivalent. This is equivalent to the statement that the This implies that the first and last sides of U are parallel. ♠

For the example above we have $((12)) = ((23)) = ((31)) = 1$. Hence, this example is stable.

There are a variety of reformulations of stability, which we will explore in other note packets.